FM 5002 Name: _____

Final Exam 11 May 2016 5:30–8:00pm

INSTRUCTIONS — PLEASE READ THIS NOW

• This test has 7 problems and 104 points, 4 of which are bonus points. Please carefully write all your final answers on the page they are posed. There is one extra blank sheet of paper at the back of this booklet for scratch work; work on this page will not be graded.

• Look over your test packet **as soon as the exam begins**. If you find any missing pages or problems please ask a proctor for another test booklet.

• **SHOW YOUR WORK.** To receive full credit, your answers must be neatly written and logically organized, with **mathematical justification**. If you need more space, write on the back side of the preceding sheet, but be sure to label your work clearly. You do not need to simplify your answers unless explicitly instructed to do so.

• You have 2.5 hours to complete this exam.

• This is a closed book exam. You may use a calculator. **One** 8.5×11 **sheet of paper with hand-written notes is allowed.** No other resources – internet, phone, etc. – are allowed!

• Academic integrity is expected of all University of Minnesota students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

Please sign below to indicate that you have read and agree to these instructions.

_____

Signature of Student

OFFICIAL USE ONLY

Question	Points	Score
1	14	
2	21	
3	23	
4	10	
5	8	
6	12	
7	16	
Total:	104	

1. Consider the function
$$f(x, y) = x^2 + 4y^2 - 2x^2y + 4.$$
(a) (8 points) Find and classify the local maxima and minima of the function.

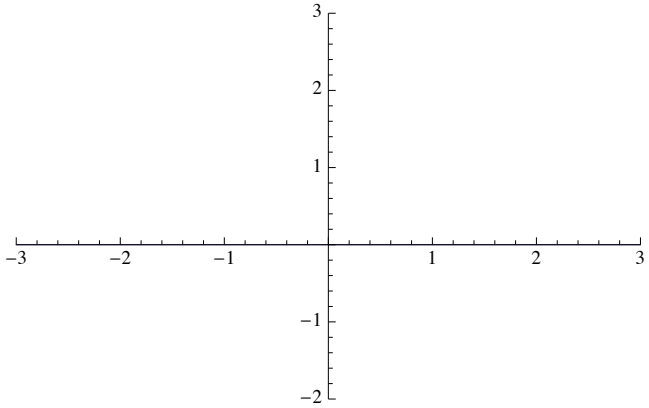
(b) (6 points) Now find the absolute maximum and minimum in the region $0 \leq t \leq 1, 0 \leq y \leq 1$.

2. Consider the differential equation

$$\frac{dy}{dt} = y^2 - t^2.$$

Notice it's non-linear and not separable.

(a) (8 points) Draw a slope field for this differential equation, with slopes drawn at all the integer points satisfying $-2 \leq t \leq 2$ and $-1 \leq y \leq 2$.



(b) (9 points) For the initial value problem with $y(0) = 0$, use Euler's method to estimate the solution at $y(1.5)$. Use a step size of 0.5.

(c) (4 points) Is your estimate an overestimate or an underestimate? Explain your answer, using the slope field above if appropriate.

3. This question will lead you to make the Markowitz portfolio optimization problem into a linear algebra problem. You won't actually solve for the optimal portfolio. You are considering two stocks A and B, with expected returns $r_A = 0.12$ and $r_B = 0.08$, and return standard deviations $\sigma_A = 1/2$ and $\sigma_B = 1/3$. You want to set up a portfolio (w_A, w_B) that puts fraction w_A of your capital into stock A and fraction w_B into stock B. Note that $w_A + w_B = 1$. The returns of the stocks have a correlation of $\rho = 0.3$.

   (a) (7 points) You want to minimize variance/volatility of the portfolio and attain an expected portfolio return of 0.1. Clearly indicating what your variables are, write equations for

   - the function you want to minimize: _____
   - the constraint coming from desired portfolio return: _____
   - the constraint coming from the construction of the weights: _____

   (b) (5 points) Write the Lagrangian function combining the above, using λ_1 and λ_2 as your Lagrange multipliers.

   (c) (4 points) Find the gradient of the Lagrangian function.

   (d) (7 points) The gradient and constraints give four linear equations. Fill in all the empty spots in the following matrix equation coming from this system of equations.

$$\begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} w_A \\ w_B \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0.1 \end{pmatrix}$$

   (e) Solving this system of equations will then give you the weights of the portfolio and the values of the Lagrange multipliers! (To confirm: this is a comment. Take no written action here.)

4. Let $\vec{X}$ be distributed as $\mathcal{N}(\vec{\mu}, \Sigma)$ where $\vec{\mu} = (1, -1)$ and

$$\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

(a) (4 points) Are X_1 and X_2 independent? Explain.

(b) (6 points) How is the random variable $W = X_1 - X_2$ distributed? Give a full characterization of the probability distribution and justify.

5. (a) (4 points) Use Newton's method to find the roots of

$$x^2 + x + 1,$$

with an initial guess of $x_0 = 2$. Show at least three steps.

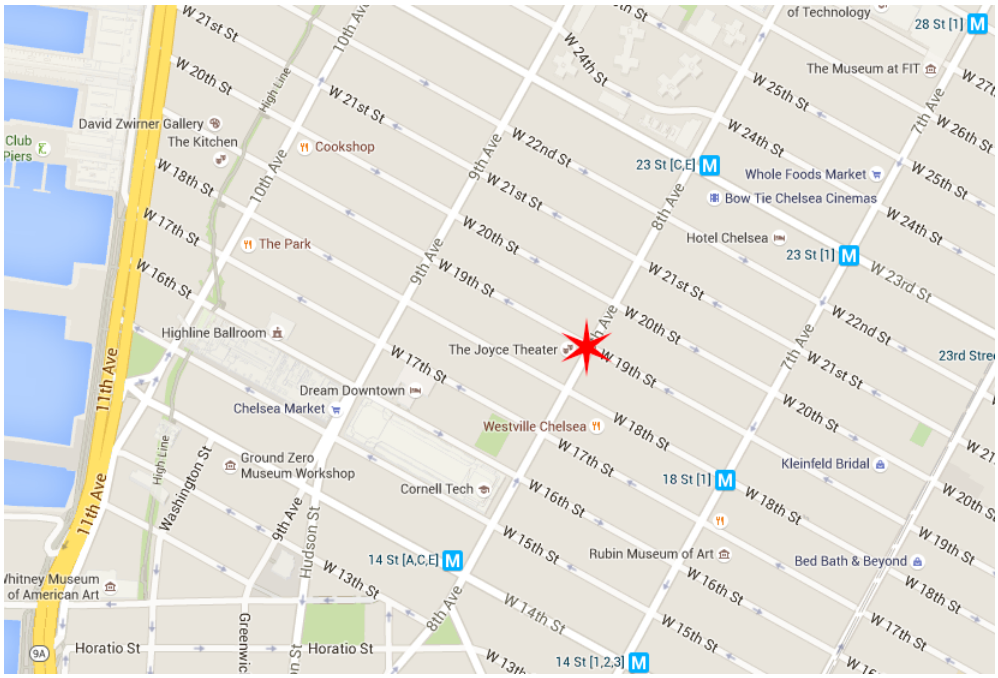
(b) (4 points) Describe what you encounter and explain it. Tell me what I need to do to successfully find the roots of the polynomial.

6. (12 points) Use a Taylor series of degree two (a quadratic approximation) to approximate $f(0.9, 2.01)$ if you know that

   - $f(1,2) = 10$
   - $\nabla f(1,2) = \begin{pmatrix} 1/2 \\ -1 \end{pmatrix}$
   - $Hf(1,2) = \begin{pmatrix} 0 & 8 \\ 8 & -2 \end{pmatrix}$
   - f is a continuous function with smooth partial derivatives.

   Show all your work.

7. Let's end with the classic drunkard's walk! Your drunkard (or sober experimenter?) is walking on the city streets of Manhattan and he's forgotten how to get to the Financial District. Let X_t be the number of blocks east or west of his starting point at time t, and let Y_t be the number of blocks north or south of his starting point at time t. We will only look at integer times t and assume he walks *one* block east or west in one unit of time, but *two* blocks north or south in one unit of time, starting at the star. Each direction has equal probability. (Example on next page.)

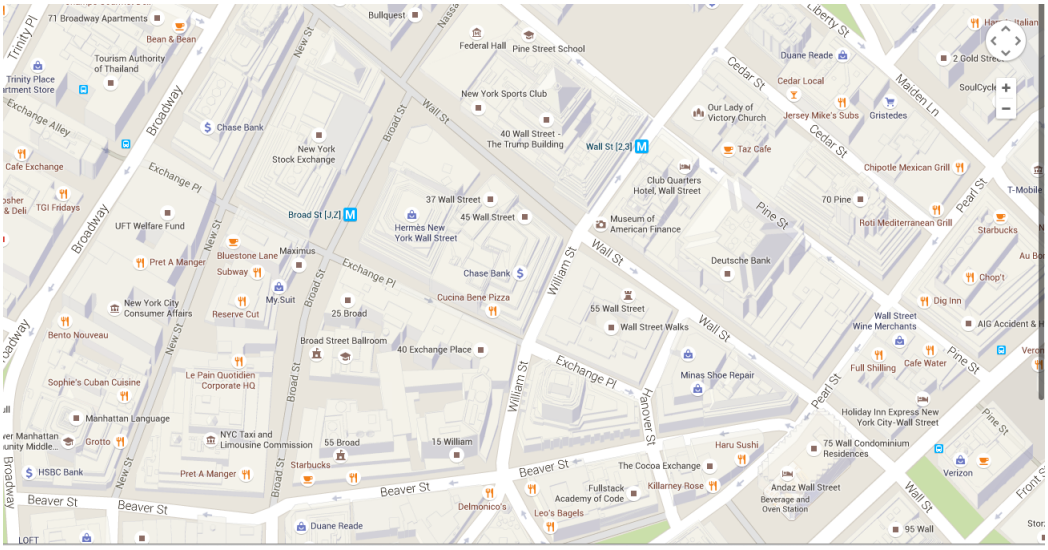


(a) (8 points) For $t = 2$, give the joint probability mass function $P(X_2 = k, Y_2 = j)$.

(b) (4 points) By $t = 4$, what problem do you encounter?

Example of a path that goes north for one time unit and east for one time unit.

(c) (4 points) Bonus question: why didn't I choose to give you a random walk down Wall Street? Explain for four bonus points!



This page is for scratch work.

Don't forget to transfer your final work to the page where the question is posed!

This page is for scratch work.

Don't forget to transfer your final work to the page where the question is posed!